

Homogeneous Functions

Consider a function involving two independent variables x and y
 $Z = f(x, y)$

① Suppose, $Z = f(x, y) = \frac{x^2}{y^2} + \frac{y^2}{x^2}$

$$f(kx, ky) = \frac{(kx)^2}{(ky)^2} + \frac{(ky)^2}{(kx)^2}$$

$$= \frac{k^2 x^2}{k^2 y^2} + \frac{k^2 y^2}{k^2 x^2}$$

$$= \frac{x^2}{y^2} + \frac{y^2}{x^2}$$

$$= k^0 \cdot f(x, y)$$

$$= f(x, y) = Z$$

② Suppose, $Z = f(x, y) = x^3 + 3x^2y + 3xy^2 + y^3$

$$f(kx, ky) = (kx)^3 + 3(kx)^2(ky) + 3(kx)(ky)^2 + (ky)^3$$

$$= k^3 x^3 + 3k^3 x^2 y + 3k^3 x y^2 + k^3 y^3$$

$$= k^3 (x^3 + 3x^2y + 3xy^2 + y^3)$$

$$= k^3 f(x, y)$$

$$= k^3 (Z)$$

\Rightarrow When inputs are changed by the same proportion k ,
the function Z changes by some power of k . If x and y has changed the function Z by
 k^0, k^3 times respectively. This property renders
these functions to be homogeneous functions.

All functions are not homogeneous.

$$z = x^2 + 2y$$

$$z = k^2x^2 + 2ky$$

$$= k(kx^2 + 2y)$$

$$\neq kz.$$

→ If it is only when the degree of homogeneity is one that the function is referred to as linearly homogeneous function.

→ A function can be linearly homogeneous yet may not be linear in relation.

$$z = f(x, y) = \sqrt{x^2 + 3xy + y^2}$$

The word "linear" here should always be interpreted only as a synonym of the first degree."

Properties of Linear Homogeneous Functions

1 - The given linear homogeneous function can be written in either of the following two forms:

$$z = x\phi\left(\frac{y}{x}\right) \text{ or } z = y\psi\left(\frac{x}{y}\right)$$

$$\text{Let } k = \frac{1}{x} \quad z = f(xy)$$

$$f(kx, ky) = k \cdot f(x, y)$$

$$\left(1, \frac{y}{x}\right) = \frac{1}{x} \cdot f(x, y)$$

$$f(x, y) = x + \left(1, \frac{y}{x}\right)$$

$$= x \cdot \phi\left(\frac{y}{x}\right)$$

$$\text{if } k = \frac{1}{y} \quad f\left(\frac{x}{y}, 1\right) = \frac{1}{y} \cdot f(x, y)$$

$$f(x, y) = y \cdot \psi\left(\frac{x}{y}\right)$$

2- The partial derivatives of the dependent variable of the given linear homogeneous functions (i.e., $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$) are functions of the ratio of x to y .

From Property I, $z = x \phi\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = \phi\left(\frac{y}{x}\right) + x \cdot \phi'\left(\frac{y}{x}\right) \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) \xrightarrow{\text{Product Rule}}$$

$$\frac{\partial z}{\partial x} = \phi\left(\frac{y}{x}\right) - \frac{y}{x} \phi'\left(\frac{y}{x}\right)$$

\therefore Partial derivative of z with respect to x is function of ratio of the independent variables.

Hence, $\frac{\partial z}{\partial x}$ is homogeneous of degree 0 in y w.r.t. x .

3- The linearly homogeneous functions satisfy "Euler's Theorem".

$$z = f(x, y)$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = z$$

Substitute values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ obtained in Property 2 in

$$\left[x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} \right]$$

$$= x \left\{ \phi\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right) \phi'\left(\frac{y}{x}\right) \right\} + y \phi'\left(\frac{y}{x}\right)$$

$$= x \cdot \phi\left(\frac{y}{x}\right) - y \cdot \phi'\left(\frac{y}{x}\right) + y \cdot \phi'\left(\frac{y}{x}\right)$$

$$= x \cdot \phi\left(\frac{y}{x}\right)$$

$$= f(x, y) = z \quad (\text{Property I})$$

Example: $Z = \sqrt[3]{x^2y}$

$$\frac{\partial Z}{\partial x} = \frac{2}{3} x^{1/2} x^{1/3} y^{1/3} \text{ and } y^{-2/3}$$

$$\frac{\partial Z}{\partial y} = \frac{1}{3} x^{2/3} y^{-2/3}$$

Therefore $x \cdot \frac{\partial Z}{\partial x} + y \cdot \frac{\partial Z}{\partial y} = x \left(\frac{2}{3} x^{1/3} y^{1/3} \right) + y \left(\frac{1}{3} x^{2/3} y^{-2/3} \right)$

$$= x^{2/3} y^{1/3}$$

$$= f(x, y) = Z$$

for Cobb-Douglas Functions

$$Q = \alpha K^\alpha L^{1-\alpha}$$

where, K is capital, L is labour and α is constant

$$K \frac{\partial Q}{\partial K} + L \frac{\partial Q}{\partial L} = Q$$

$$\frac{\partial Q}{\partial K} = \alpha^2 K^{\alpha-1} L^{1-\alpha}$$

$$\frac{\partial Q}{\partial L} = \alpha(1-\alpha) K^\alpha L^\alpha$$

Therefore, $K \frac{\partial Q}{\partial K} + L \frac{\partial Q}{\partial L} = K(\alpha^2 K^{\alpha-1} L^{1-\alpha}) + L[\alpha(1-\alpha) K^\alpha L^\alpha]$

$$= \alpha^2 K^\alpha L^{1-\alpha} + \alpha(1-\alpha) K^\alpha L^{1-\alpha}$$

$$= \alpha K^\alpha L^{1-\alpha} [\alpha + (1-\alpha)]$$

$= Q$ hence proved.

$$\therefore \frac{\partial Q}{\partial K} = \alpha K^{\alpha-1} L^{1-\alpha}$$

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(not necessary) $[x_3 - x_3 \frac{b^2}{a^2}] \frac{d^2g}{dx^2} = ab^2g \quad (3)$